Sec. 11.3 The Short Run Behavior of Polynomials

Ex: Compare the graphs of the polynomials f, g, and h given by $f(x) = x^4 - 4x^3 + 16x - 16$, $g(x) = x^4 - 4x^3 - 4x^2 + 16x$, $h(x) = x^4 + x^3 - 8x^2 - 12x$. What do you notice about the graphs? (How are they alike and how are they different?)

$$f(x) = x^4 - 4x^3 + 16x - 16, \quad g(x) = x^4 - 4x^3 - 4x^2 + 16x, \quad h(x) = x^4 + x^3 - 8x^2 - 12x.$$

Each of these functions is a fourth-degree polynomial, and each has x^4 as its leading term. Thus, all their graphs resemble the graph of x^4 on a large scale.

Figure 11.2

However, on a smaller scale, the functions look different:

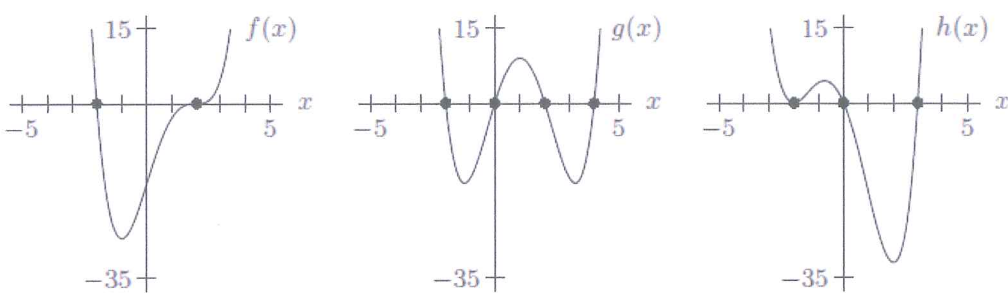


Figure 11.3

To predict the long-run behavior of a polynomial, we use the highest-power term. To determine the zeros and the short-run behavior of a polynomial, we write it in factored form with as many linear factors as possible.

Ex: Investigate the short-run behavior of the third degree polynomial $u(x) = x^3 - x^2 - 6x$.

(a) Rewrite $u(x)$ as a product of linear factors.

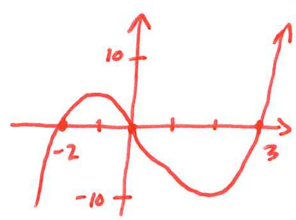
$x(x^2 - x - 6) = x(x-3)(x+2)$

(b) Find the zeros of $u(x)$. $0 = x(x-3)(x+2)$

$x = 0 \quad x - 3 = 0 \quad x + 2 = 0$

$x = 3 \quad x = -2$

(c) Describe the graph of $u(x)$. Where does it cross the x-axis? The y-axis? Where is $u(x)$ positive? Negative? The graph resembles $y = x^3$ in the long run. It crosses the x-axis at $x = -2$, $x = 0$, and $x = 3$. It crosses the y-axis at $u(0) = 0^3 - 0^2 - 6(0) = 0$.



$u(x)$ is positive $-2 < x < 0 \quad x > 3$

$u(x)$ is negative $x < -2 \quad 0 < x < 3$

Suppose p is a polynomial:

- If the formula for p has a **linear factor**, that is, a factor of the form $(x - k)$, then p has a zero at $x = k$.
- Conversely, if p has a **zero** at $x = k$, then p has a linear factor of the form $(x - k)$.
- The graph of an nth degree polynomial has at most n zeros and turns at most $(n - 1)$ times.

If p is a polynomial with a repeated linear factor, then p has a ***multiple zero****.*

- If the factor $(x - k)$ occurs an even number of times, the graph of $y = p(x)$ does not cross the x-axis at $x = k$, but **"bounces" off** the x-axis at $x = k$.
- If the factor $(x - k)$ occurs an odd number of times, the graph of $y = p(x)$ crosses the x-axis at $x = k$, but it looks **flattened** there.

Ex: Given the equation $f(x) = (x-5)^2 (x+2)^5 (x-4)^6$, what are the zeros, their multiplicities, the end behavior and does the graph touch or cross at each zero?

$0 = (x-5)^2 (x+2)^2 (x-4)^6$

$x-5=0$ $\quad x+2=0$ $\quad x-4=0$

$x=5$ $\quad x=-2$ $\quad x=4$

(multiplicity 2) (mult 5) (mult. 6)

touch cross touch

End behavior will resemble graph of $y=x^6$. As $x \to \pm\infty$, $y \to \infty$

Ex: Identify any multiple zeros in figure 11.3 and write possible factored forms for each.

Multiple zero at $x=2$
{flat there}
$f(x) = (x+2)(x-2)^3$

No multiple zeros
$g(x) = (x+2)x(x-2)(x-4)$
$g(x) = x(x+2)(x-2)(x-4)$

Multiple zero at $x=-2$
{bounces off}
$h(x) = (x+2)^2 x(x-3)$
$h(x) = x(x+2)^2(x-3)$

Ex: Find the polynomial of degree 6 whose zeros are 3 (multiplicity 2), 2 (multiplicity 3), and –1 and has a y-intercept of 5.

$y = k(x-3)^2(x-2)^3(x+1)$

$5 = k(0-3)^2(0-2)^3(0+1)$

$5 = k(9)(-8)(1)$

$5 = -72k$

$-\frac{5}{72} = k$

$y = -\frac{5}{72}(x-3)^2(x-2)^3(x+1)$

Ex: For the polynomial $f(x) = 4(x+3)^2(x-7)^4(x+1/2)^3$, find the following:

a. Find the x and y-intercepts of the graph of f.
b. Using a graphing calculator, graph the polynomial.
c. For each x-intercept, determine if it has even or odd multiplicity.

$0 = 4(x+3)^2(x-7)^4(x+\frac{1}{2})^3$

$x+3=0$ $\quad x-7=0$ $\quad x+\frac{1}{2}=0$

$x=-3$ (EVEN) $\quad x=7$ (EVEN) $\quad x=-\frac{1}{2}$ (ODD)

WINDOW $-4 \le x \le 8$

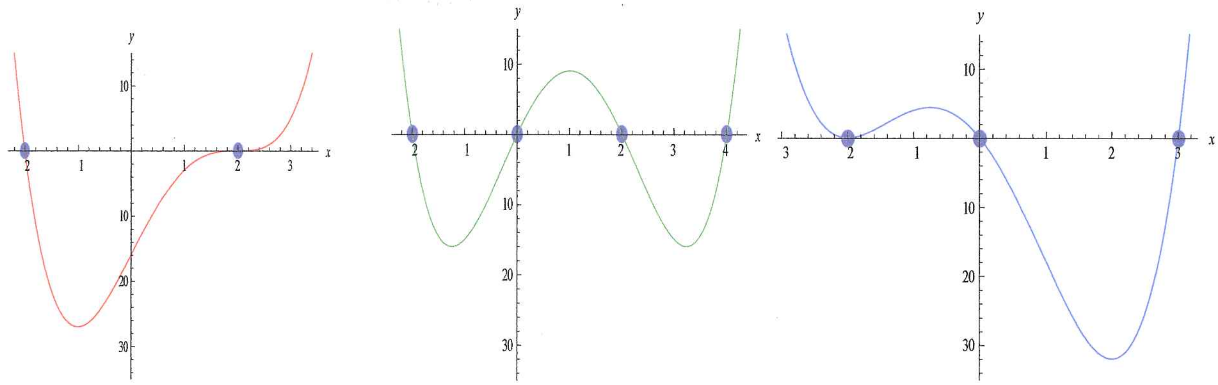
$-200{,}000 \le y \le 2{,}000{,}000$

$f(0) = 4(0+3)^2(0-7)^4(0+\frac{1}{2})^3$

$= 4(9)(2401)(\frac{1}{8})$

$f(0) = 10{,}804.5$

Ex: Describe in words the zeros of the 4th-degree polynomials $f(x)$, $g(x)$, and $h(x)$, in the graphs below and find a formula for each.



Single-zero at $x=-2$, multiple zero at $x=2$ {flattened} - must be odd, so triple zero there.

$f(x)=K(x+2)(x-2)^3$

$-16=K(0+2)(0-2)^3$

$-16=K(-16)$

$1=K$

$\boxed{f(x)=(x+2)(x-2)^3}$

Single zeros at $x=-2,0,2,4$

$g(x)=(x+2)(x)(x-2)(x-4)$

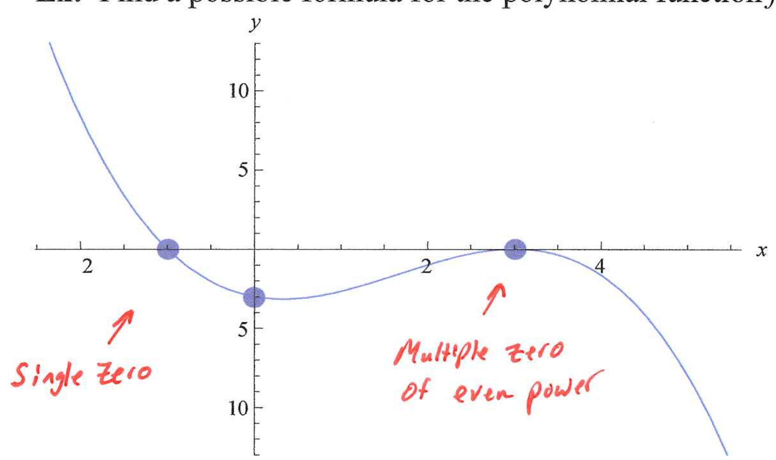
$\boxed{g(x)=x(x+2)(x-2)(x-4)}$

Double zero at $x=-2$ {bounces off}. Single at $x=0$ and $x=3$.

$h(x)=(x+2)^2(x)(x-3)$

$\boxed{h(x)=x(x+2)^2(x-3)}$

Ex: Find a possible formula for the polynomial function f graphed below.



$f(x)=K(x+1)(x-3)^2$

$-3=K(0+1)(0-3)^2$

$-3=9K$

$-\frac{1}{3}=K$

$\boxed{f(x)=-\frac{1}{3}(x+1)(x-3)^2}$

or

$f(x)=K(x+1)(x-3)^4$

$-3=K(0+1)(0-3)^4$

$-3=81K$

$-\frac{1}{27}=K$

$\boxed{f(x)=-\frac{1}{27}(x+1)(x-3)^4}$

HW: pg 452-454 #1,4-7,9,10,14,15,19,23-25,28,29,35,47